Exercise 37

Solve the initial-value problem in Exercise 9.2.27 to find an expression for the charge at time t. Find the limiting value of the charge.

Solution

The initial-value problem in Exercise 9.2.27 is the following:

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t), \qquad Q(0) = 0,$$

where $R = 5\ \Omega$, $E(t) = 60$ V, and $C = 0.05$ F.

$$\begin{aligned} 5\frac{dQ}{dt} + 20Q &= 60 \\ \frac{dQ}{dt} + 4Q &= 12 \\ \frac{dQ}{dt} &= 12 - 4Q. \end{aligned}$$

This is a separable equation, so we can solve for $Q(t)$ by bringing all terms with Q to the left and all constants and terms with t to the right and then integrating both sides.

$$\begin{aligned} dQ &= 4(3 - Q)\,dt \\ \frac{dQ}{3 - Q} &= 4\,dt \\ \int \frac{dQ}{3 - Q} &= \int 4\,dt \end{aligned}$$

Use a u-substitution to solve the integral on the left.

$$\begin{aligned} \text{Let } u &= 3 - Q \\ du &= -dQ \quad \rightarrow \quad dQ = -du \end{aligned}$$

$$\begin{aligned} \int \frac{-du}{u} &= \int 4\,dt \\ -\ln|u| &= 4t + D \\ \ln|3 - Q| &= -4t - D \\ e^{\ln|3-Q|} &= e^{-4t-D} \\ |3 - Q| &= e^{-4t}e^{-D} \\ 3 - Q &= \pm e^{-D}e^{-4t} \end{aligned}$$

Let $D_1 = \pm e^{-D}$.

$$Q(t) = 3 - D_1 e^{-4t}$$

We can use the initial condition to determine D_1.

$$\begin{aligned} Q(0) = 3 - D_1 &= 0 \\ D_1 &= 3 \end{aligned}$$

Therefore,

$$Q(t) = 3\left(1 - e^{-4t}\right).$$

The limiting value of the charge is found by taking the limit as t goes to infinity.

$$\begin{aligned}
\lim_{t\to\infty} Q(t) &= \lim_{t\to\infty} 3\left(1 - e^{-4t}\right) \\
&= \lim_{t\to\infty} 3 - \lim_{t\to\infty} 3e^{-4t} \\
&= \lim_{t\to\infty} 3 - 3\underbrace{\lim_{t\to\infty} e^{-4t}}_{=0} \\
&= 3 - 3(0)
\end{aligned}$$

Thus,

$$\lim_{t\to\infty} Q(t) = 3 \text{ C}.$$